MAT1332 Calculus for Life Science II

Assignment #4

Due date: March 11

Question 1:
Solve the following systems of linear equations:

a)
$$\begin{aligned} x + 3y + z &= 4 \\ 2x + 3y - 3z &= 5 \\ 3x + 5y - 5z &= 0 \end{aligned}$$

b)
$$\begin{aligned} 2x + 2y - z &= -5 \\ 3x + y - 3z &= 3 \\ -4x + 5z &= -11 \end{aligned}$$

Solution:
a) The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} 1 & 3 & 1 & 4 \\ 2 & 3 & -3 & 5 \\ 3 & 5 & -5 & 0 \end{array}\right].$$

$R_2 - 2R_1 \to R_2$ and $R_3 - 3R_1 \to R_3$ give

$$\left[\begin{array}{rrr|r} 1 & 3 & 1 & 4 \\ 0 & -3 & -5 & -3 \\ 0 & -4 & -8 & -12 \end{array}\right].$$

$(-1/3)R_2 \to R_2$ and $(-1/4)R_3 \to R_3$ give

$$\left[\begin{array}{rrr|r} 1 & 3 & 1 & 4 \\ 0 & 1 & 5/3 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right].$$

$R_1 - 3R_2 \to R_1$ and $R_3 - R_2 \to R_3$ give

$$\left[\begin{array}{rrr|r} 1 & 0 & -4 & 1 \\ 0 & 1 & 5/3 & 1 \\ 0 & 0 & 1/3 & 2 \end{array}\right].$$

$3R_3 \to R_3$ gives

$$\left[\begin{array}{rrr|r} 1 & 0 & -4 & 1 \\ 0 & 1 & 5/3 & 1 \\ 0 & 0 & 1 & 6 \end{array}\right].$$

$R_1 + 4R_3 \to R_1$ and $R_2 - (5/3)R_3 \to R_2$ give

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 25 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 6 \end{array}\right].$$

We get $z = 6$, $y = -9$ and $x = 25$.

b) The augmented matrix is

$$\left[\begin{array}{rrr|r} 2 & 2 & -1 & -5 \\ 3 & 1 & -3 & 3 \\ -4 & 0 & 5 & -11 \end{array}\right].$$

$R_2 - R_1 \to R_2$ and $R_3 + 2R_1 \to R_3$ give

$$\left[\begin{array}{rrr|r} 2 & 2 & -1 & -5 \\ 1 & -1 & -2 & 8 \\ 0 & 4 & 3 & -21 \end{array}\right].$$

$R_1 - 2R_2 \to R_1$ gives

$$\left[\begin{array}{rrr|r} 0 & 4 & 3 & -21 \\ 1 & -1 & -2 & 8 \\ 0 & 4 & 3 & -21 \end{array}\right].$$

$R_3 - R_1 \to R_3$ gives

$$\left[\begin{array}{rrr|r} 0 & 4 & 3 & -21 \\ 1 & -1 & -2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$(1/4)R_1 \to R_1$ gives

$$\left[\begin{array}{rrr|r} 0 & 1 & 3/4 & -21/4 \\ 1 & -1 & -2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$R_2 + R_1 \to R_2$ gives

$$\left[\begin{array}{rrr|r} 0 & 1 & 3/4 & -21/4 \\ 1 & 0 & -5/4 & 11/4 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, $x = 11/4 + 5\alpha/4$, $y = -21/4 - 3\alpha/4\alpha$ and $z = \alpha \in \mathbb{R}$.

Question 2:

Consider the system of linear equations

$$\begin{aligned} x + ay &= 2 \\ 2x + 5y &= b \end{aligned}$$

Find the values of a and b such that the system has:

a) A unique solution.

b) An infinite number of solutions.

c) No solution.

Solution:

The augmented matrix of the system is

$$\left[\begin{array}{cc|c} 1 & a & 2 \\ 2 & 5 & b \end{array}\right].$$

$R_2 - 2R_1 \to R_2$ gives

$$\left[\begin{array}{cc|c} 1 & a & 2 \\ 0 & 5-2a & b-4 \end{array}\right]. \tag{1}$$

a) If $5-2a \neq 0$, there is a unique solution. We can perform the operation $\left[\dfrac{1}{5-2a}\right] R_2 \to R_2$ on (1) to get

$$\left[\begin{array}{cc|c} 1 & a & 2 \\ 0 & 1 & (b-4)/(5-2a) \end{array}\right].$$

$R_1 - aR_2 \to R_2$ gives

$$\left[\begin{array}{cc|c} 1 & 0 & (10-ab)/(5-2a) \\ 0 & 1 & (b-4)/(5-2a) \end{array}\right].$$

Thus, $x = (10-ab)/(5-2a)$ and $y = (b-4)/(5-2a)$.

b) If $5-2a = 0$ and $b-4 = 0$, then the system (1) is deduced to

$$\left[\begin{array}{cc|c} 1 & a & 2 \\ 0 & 0 & 0 \end{array}\right].$$

There are infinitely many solutions given by $x = 2 - a\alpha$ and $y = \alpha \in \mathbb{R}$.

c) If $5-2a = 0$ and $b-4 \neq 0$, then there is no solution because the last row of the system (1) is $0 = b-4 \neq 0$.

Question 3:
Consider the matrices

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 3 \\ -2 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}, \ C = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -1 & 5 \end{bmatrix},$$
$$D = \begin{bmatrix} 1 & 4 & 2 \\ -2 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} \text{ and } E = \begin{bmatrix} 3 & 1 & 3 \\ -1 & 1 & -1 \\ -2 & 1 & 3 \end{bmatrix}.$$

Evaluate if possible the following algebraic expressions.

a)	$2B - C$	**b)**	$4E - 2D$	**c)**	$2A^\top + C$
d)	$B^\top + 5C^\top$	**e)**	$A(BC)$	**f)**	$E^\top D^\top$
g)	$(DE)^\top$				

Solution:

a) The operation $2B - C$ is not defined because the matrices $2B$ and C don't have the same dimensions.

b)

$$4E - 2D = \begin{bmatrix} 10 & -4 & 8 \\ 0 & 4 & -6 \\ -14 & 0 & 14 \end{bmatrix}.$$

c)

$$2A^\top + C = 2\begin{bmatrix}3 & 1\\ -1 & 3\\ -2 & 1\end{bmatrix}^\top + \begin{bmatrix}-1 & 4 & 2\\ 3 & -1 & 5\end{bmatrix}$$

$$= 2\begin{bmatrix}3 & -1 & -2\\ 1 & 3 & 1\end{bmatrix} + \begin{bmatrix}-1 & 4 & 2\\ 3 & -1 & 5\end{bmatrix} = \begin{bmatrix}5 & 2 & -2\\ 5 & 5 & 7\end{bmatrix}.$$

d) The operation $B^\top + 5C^\top$ is not defined because the matrices $B^\top$ and $5C^\top$ don't have the same dimensions.

e)

$$A(BC) = \begin{bmatrix}3 & 1\\ -1 & 3\\ -2 & 1\end{bmatrix}\left[\begin{bmatrix}3 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}-1 & 4 & 2\\ 3 & -1 & 5\end{bmatrix}\right]$$

$$= \begin{bmatrix}3 & 1\\ -1 & 3\\ -2 & 1\end{bmatrix}\begin{bmatrix}-6 & 13 & 1\\ 5 & 2 & 12\end{bmatrix} = \begin{bmatrix}-13 & 41 & 15\\ 21 & -7 & 35\\ 17 & -24 & 10\end{bmatrix}$$

f)

$$E^\top D^\top = \begin{bmatrix}3 & 1 & 3\\ -1 & 1 & -1\\ -2 & 1 & 3\end{bmatrix}^\top \begin{bmatrix}1 & 4 & 2\\ -2 & 0 & 1\\ 3 & 2 & -1\end{bmatrix}^\top$$

$$= \begin{bmatrix}3 & -1 & -2\\ 1 & 1 & 1\\ 3 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & -2 & 3\\ 4 & 0 & 2\\ 2 & 1 & -1\end{bmatrix} = \begin{bmatrix}-5 & -8 & 9\\ 7 & -1 & 4\\ 5 & -3 & 4\end{bmatrix}.$$

g)

$$(DE)^\top = E^\top D^\top = \begin{bmatrix}-5 & -8 & 9\\ 7 & -1 & 4\\ 5 & -3 & 4\end{bmatrix}.$$

Question 4:
For each of the matrices below, determine if the matrix is invertible and, if it is invertible, find its inverse.

$$\textbf{a)} \quad A = \begin{bmatrix}1 & 1 & 1\\ 2 & -2 & 0\\ 2 & 1 & 3\end{bmatrix} \qquad \textbf{b)} \quad B = \begin{bmatrix}1 & 2 & 3\\ 2 & 1 & -1\\ -1 & 1 & 4\end{bmatrix}$$

a) If we expand along the second row, we get

$$\det(A) = -2\det\begin{bmatrix}1 & 1\\ 1 & 3\end{bmatrix} - 2\det\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix} = -2(3-1) - 2(3-2) = -6 \neq 0\,.$$

Since the determinant of the matrix A is not 0, the matrix A is invertible. Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right].$$

$R_2 - 2R_1 \to R_2$ and $R_3 - 2\,R_1 \to R_3$ give

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & -2 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 \end{array}\right].$$

$-R_3 \to R_3$ fallowed by $R_2 \leftrightarrow R_3$ give

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & -1 \\ 0 & -4 & -2 & -2 & 1 & 0 \end{array}\right].$$

$R_3 + 4R_2 \to R_2$ and $R_1 - R_2 \to R_1$ give

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 & 0 & -1 \\ 0 & 0 & -6 & 6 & 1 & -4 \end{array}\right].$$

$(-1/6)R_3 \to R_3$ gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1/6 & 2/3 \end{array}\right].$$

$R_1 - 2\,R_3 \to R_1$ and $R_2 + R_3 \to R_2$ give

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1/3 & -1/3 \\ 0 & 1 & 0 & 1 & -1/6 & -1/3 \\ 0 & 0 & 1 & -1 & -1/6 & 2/3 \end{array}\right].$$

We find

$$A^{-1} = \begin{bmatrix} 1 & 1/3 & -1/3 \\ 1 & -1/6 & -1/3 \\ -1 & -1/6 & 2/3 \end{bmatrix}.$$

b) If we expand along the first column, we have

$$\det(B) = \det \begin{bmatrix} 2 & 8 \\ 3 & 7 \end{bmatrix} + 2\det \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} = (14-24) + 2(14-9) = 0\,.$$

Therefore, the matrix B is not invertible.

Question 5:
The food of laboratory mice must contain the nutrients A and B. The mice are fed two

types of food. The first type contains 3 units of A and 2 units of B, and the second type contains 4 units of A and 5 units of B per gramme.

a) How many grams of each type of food must we have to get exactly 125 units of A and 100 units of B?

b) Find the ratio between the number of grams of the two types of food if the ratio of the number of units of A over the number of units of B is 4 to 3.

c) Can we get a ratio of 2 to 1 for the number of units of A over the number of units of B?

Solution:
Let x be the number of grams of the first type of food and y be the number of grams of the second type of food. The number of units of A given to the mice is $3x + 4y$ and the number of units of B given to the mice is $2x + 5y$.

a) We must solve the system

$$\begin{aligned} 3x + 4y &= 125 \\ 2x + 5y &= 100 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{cc|c} 3 & 4 & 125 \\ 2 & 5 & 100 \end{array}\right].$$

$R_1 - R_2 \to R_1$ gives

$$\left[\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 5 & 100 \end{array}\right].$$

$R_2 - 2R_1 \to R_2$ gives

$$\left[\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 7 & 50 \end{array}\right].$$

$(1/7)R_2 \to R_2$ gives

$$\left[\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & 50/7 \end{array}\right].$$

$R_1 + R_2 \to R_1$ gives

$$\left[\begin{array}{cc|c} 1 & 0 & 225/7 \\ 0 & 1 & 50/7 \end{array}\right].$$

Thus, $x = 225/7$ g and $y = 50/7 \approx 7.143$ g.

b) We must solve the system

$$\begin{aligned} 3x + 4y &= 4a \\ 2x + 5y &= 3a \end{aligned}$$

where a is a fixed, unknown constant. The augmented matrix of the system is

$$\left[\begin{array}{cc|c} 3 & 4 & 4a \\ 2 & 5 & 3a \end{array}\right].$$

$R_1 - R_2 \to R_1$ gives

$$\left[\begin{array}{cc|c} 1 & -1 & a \\ 2 & 5 & 3a \end{array}\right] .$$

$R_2 - 2R_1 \to R_2$ gives

$$\left[\begin{array}{cc|c} 1 & -1 & a \\ 0 & 7 & a \end{array}\right] .$$

$(1/7)R_2 \to R_2$ gives

$$\left[\begin{array}{cc|c} 1 & -1 & a \\ 0 & 1 & a/7 \end{array}\right] .$$

$R_1 + R_2 \to R_1$ gives

$$\left[\begin{array}{cc|c} 1 & 0 & 8a/7 \\ 0 & 1 & a/7 \end{array}\right] .$$

Thus, $x = 8a/7$ g and $y = a/7$ g. Moreover,

$$\frac{x}{y} = \frac{8a/7}{a/7} = 8 .$$

For each gram of the second type of food, there must be 8 grams of the first type of food.

c) We must solve

$$\begin{aligned} 3x + 4y &= 2a \\ 2x + 5y &= a \end{aligned}$$

where a is again a fixed, unknown constant. The augmented matrix of the system is

$$\left[\begin{array}{cc|c} 3 & 4 & 2a \\ 2 & 5 & a \end{array}\right] .$$

$R_1 - R_2 \to R_1$ gives

$$\left[\begin{array}{cc|c} 1 & -1 & a \\ 2 & 5 & a \end{array}\right] .$$

$R_2 - 2R_1 \to R_2$ gives

$$\left[\begin{array}{cc|c} 1 & -1 & a \\ 0 & 7 & -a \end{array}\right] .$$

$(1/7)R_2 \to R_2$ gives

$$\left[\begin{array}{cc|c} 1 & -1 & a \\ 0 & 1 & -a/7 \end{array}\right] .$$

Thus, $y = -a/7 < 0$. It is not possible to have a negative number of grams. We therefore can not have a ratio of 2 to 1.